§ Submanifold theory

Goal: Generalize the classical theory for surfaces in $\mathbb{R}^3$ to submanifolds $\Sigma^k \subseteq (M^n, g)$

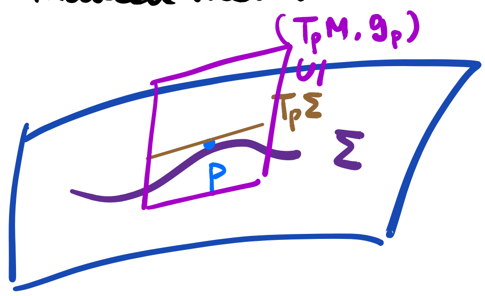
Riem. manifold

Def$^{\underline{n}}$: An isometric immersion $F : (\Sigma^k, h) \to (M^n, g)$ is an immersion (as manifolds) s.t. $F^* g = h$

As far as local aspects are concerned, we can regard

$\Sigma^k \subseteq (M^n, g)$ and $g|_\Sigma$ induced metric

as a k-dim'l embedded submfd



(M^n, g)

∇^M

Note: $(\Sigma^k, g|_\Sigma)$ Riem mfd $\xrightarrow[\text{of R.G.}]{\text{Fund. Thm}}$ $\exists !$ Riem connection ∇^Σ on Σ

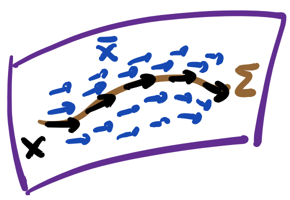
Q: How are the connections ∇^M and ∇^Σ related ?

Recall: $\exists$ "canonical" orthogonal splitting : at each $p \in \Sigma$

$$T_pM = T_p\Sigma \oplus \underbrace{(T_p\Sigma)^\perp}_{}$$

w.r.t. g — normal bundle $N_p\Sigma$

$$\begin{matrix} \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ v & = & v^T & + & v^N \end{matrix}$$



(M^n, g)

Thm: Let $X, Y \in \Gamma(T\Sigma)$. Then

$$\nabla^\Sigma_X Y = \left(\nabla^M_{\bar{X}} \bar{Y}\right)^T$$

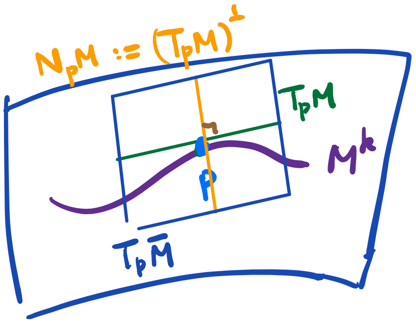
where $\bar{X}, \bar{Y} \in \Gamma(TM)$ are "extensions" of X, Y s.t. $\bar{X}|_\Sigma = X$, $\bar{Y}|_\Sigma = Y$

Recall: Isometric immersion $F : (M^k, g) \to (\bar{M}^n, \bar{g})$ s.t. $F^*\bar{g} = g$

i.e. $\bar{g}_{F(p)}(dF_p(v), dF_p(w)) = g_p(v, w)$

Locally, immersions are embedding. $M^k \approx F(M) \subseteq \bar{M}^n$.

Setup: $M^k \subseteq (\bar{M}^n, \bar{g})$ submanifold $(k \leq n)$



Question: How do study the "intrinsic" & "extrinsic" geometry of M?

Crucial observation: At $p \in M$, there is an orthogonal splitting (w.r.t. $\bar{g}$)

$$T_p\bar{M} = T_pM \oplus_\perp (T_pM)^\perp$$

Notation: $NM := \coprod_{p \in M} (T_pM)^\perp$ normal bundle

orthogonal decomposition

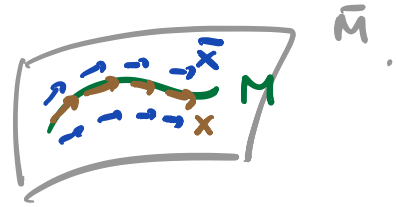
$$\underset{T_p\bar{M}}{\underset{\cap}{V}} = \underset{T_pM}{\underset{\cap}{V^T}} + \underset{N_pM}{\underset{\cap}{V^N}} \qquad \forall p \in M$$

Note: $\bar{g}$ restricts to an inner product on each $T_pM \subseteq (T_p\bar{M}, \bar{g})$

write $g := \bar{g}|_{TM} \rightsquigarrow (M^k, g)$ Riem. manifold

By Fund. Thm. of R.G., $\exists!$ Riem. connection ∇ for (M^k, g).

Q: How is ∇ related to the ambient Riem. connection $\bar{\nabla}$ on $(\bar{M}, \bar{g})$?



<u>Recall:</u> $\nabla : \mathcal{T}(TM) \times \mathcal{T}(TM) \to \mathcal{T}(TM)$

$\bar{\nabla} : \mathcal{T}(T\bar{M}) \times \mathcal{T}(T\bar{M}) \to \mathcal{T}(T\bar{M})$

<u>Prop</u>: Let $X, Y \in \mathcal{T}(TM)$, and $\bar{X}, \bar{Y} \in \mathcal{T}(T\bar{M})$ be extensions of X, Y.

THEN:

$$\boxed{\nabla_X Y = (\bar{\nabla}_{\bar{X}} \bar{Y})^T}$$

<u>Remarks:</u> $\bar{\nabla}_{\bar{X}} \bar{Y}(p)$ depends only on $\bar{X}(p) = X(p)$

and $\bar{Y}$ along any curve $\gamma : (-\varepsilon, \varepsilon) \to \bar{M}$ s.t. $\gamma(0) = p$, $\gamma'(0) = X(p)$

[we can make $\gamma \subseteq M$, where $\bar{Y} = Y$ on M]

$\Rightarrow$ R.H.S. is indep. of the choice of extensions $\bar{X}, \bar{Y}$.

<u>Proof</u>: Check R.H.S. defines a connection which is metric compatible and torsion-free, then result follows by uniqueness part of Fund Thm of R.G.

(i) $(X, Y) \mapsto (\bar{\nabla}_X Y)^T$ bilinear

(ii) $(\bar{\nabla}_{fX} Y)^T = (f \bar{\nabla}_X Y)^T = f (\bar{\nabla}_X Y)^T$

(iii) $(\bar{\nabla}_X (fY))^T = (X(f) Y + f \bar{\nabla}_X Y)^T = X(f) Y + f (\bar{\nabla}_X Y)^T$

Connection.

(iv) $Z(g(X,Y)) = \bar{Z}(\bar{g}(\bar{X}, \bar{Y})) = \bar{g}(\bar{\nabla}_{\bar{Z}} \bar{X}, Y) + \bar{g}(X, \bar{\nabla}_{\bar{Z}} \bar{Y})$

$= g((\bar{\nabla}_{\bar{Z}} \bar{X})^T, Y) + g(X, (\bar{\nabla}_{\bar{Z}} \bar{Y})^T)$

metric-compatible

(v) $(\bar{\nabla}_{\bar{X}} \bar{Y})^T - (\bar{\nabla}_{\bar{Y}} \bar{X})^T = (\bar{\nabla}_{\bar{X}} \bar{Y} - \bar{\nabla}_{\bar{Y}} \bar{X})^T = ([\bar{X}, \bar{Y}])^T$

$= ([X, Y])^T = [X, Y]$

torsion-free

$\blacksquare$

Q: What about the "normal" part of $\bar{\nabla}$?

Def^{n}: 2^{nd} fundamental form of M in $\bar{M}$

$$A : \mathcal{T}(TM) \times \mathcal{T}(TM) \longrightarrow \mathcal{T}(NM)$$

$$A(X,Y) := (\bar{\nabla}_{\bar{X}} \bar{Y})^N$$

Remark: This is well-defined indep. of the extensions $\bar{X}, \bar{Y}$.

Lemma: (i) $A(X,Y) = A(Y,X)$ $\quad \forall\, X,Y \in \mathcal{T}(TM)$, $\forall\, f \in C^\infty(M)$

(ii) $A(fX,Y) = A(X,fY) = f\, A(X,Y)$

i.e. A is a symmetric NM-valued $(0,2)$-tensor.

Proof: (i) $A(X,Y) - A(Y,X) = (\bar{\nabla}_{\bar{X}} \bar{Y})^N - (\bar{\nabla}_{\bar{Y}} \bar{X})^N$

$$= (\bar{\nabla}_{\bar{X}} \bar{Y} - \bar{\nabla}_{\bar{Y}} \bar{X})^N = ([\bar{X},\bar{Y}])^N = ([X,Y])^N = 0$$

(Because $[X,Y] \in \mathcal{T}(TM)$ $\forall\, X,Y \in \mathcal{T}(TM)$).

(ii) $A(fX,Y) = (\bar{\nabla}_{\bar{f}\bar{X}} \bar{Y})^N = (\bar{f}\, \bar{\nabla}_{\bar{X}} \bar{Y})^N = f (\bar{\nabla}_{\bar{X}} \bar{Y})^N = f\, A(X,Y)$.

$\square$

Fix $\eta \in \mathcal{T}(NM)$, then we can define a scalar-valued 2^{nd} f.f. (w.r.t η)

$$A_\eta : \mathcal{T}(TM) \times \mathcal{T}(TM) \longrightarrow C^\infty(M)$$

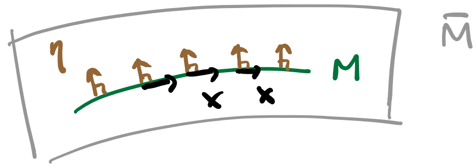
$$A_\eta(X,Y) = \langle A(X,Y), \eta \rangle$$

This is a symmetric bilinear form on each T_pM.

linear algebra $\rightsquigarrow$ $\quad A_\eta(X,Y) = \langle S_\eta(X), Y \rangle$ $\quad$ shape operator / Weingarten map

where $S_\eta : T_pM \to T_pM$ is self-adjoint operator.

Prop: $\boxed{S_\eta(X) = -(\bar{\nabla}_X \eta)^T}$



Proof: $\langle S_\eta(X), Y \rangle = A_\eta(X,Y) = \langle A(X,Y), \eta \rangle$ $\quad \forall X,Y \in \Gamma(TM)$

$$= \langle (\bar{\nabla}_{\bar{X}} \bar{Y})^N, \eta \rangle = \langle \bar{\nabla}_{\bar{X}} \bar{Y}, \eta \rangle$$

$$= X \underbrace{\langle Y, \eta \rangle}_{\equiv 0} - \langle Y, \bar{\nabla}_X \eta \rangle = \langle Y, -(\bar{\nabla}_X \eta)^T \rangle$$

$\square$

Question:

$(\bar{M}^n, \bar{g}) \rightsquigarrow$ connection $\bar{\nabla} \rightsquigarrow$ curvature $\bar{R}$

$\cup$ $\qquad$ related $\updownarrow$ $\qquad$ $\updownarrow$ relation?

$(M^k, g) \rightsquigarrow$ connection $\nabla \rightsquigarrow$ curvature R

Answer: $\bar{R}$ and R are related via the 2^{nd} f.f. A (or S_η)

They will be expressed in terms of 3 sets of "constraint equations" called Gauss, Codazzi, Ricci equations.

Before we state these equations, we need some preliminary notions:

(i) Connection & curvature on normal bundle NM

$\exists$ connection $\nabla^\perp$ on NM defined as:

$$\nabla^\perp : \Gamma(TM) \times \Gamma(NM) \to \Gamma(NM)$$

$$\nabla^\perp_X \eta := (\bar{\nabla}_X \eta)^N$$

(Ex: This is a "connection")

$\rightsquigarrow$ normal curvature $\quad R^\perp(X,Y)\eta := \nabla^\perp_Y \nabla^\perp_X \eta - \nabla^\perp_X \nabla^\perp_Y \eta + \nabla^\perp_{[X,Y]} \eta$

(ii) Covariant derivative of 2^{nd} f.f. A

Fix $\eta \in \Gamma(NM)$, then

$$A_\eta(X,Y) = \langle A(X,Y), \eta \rangle =: A(X,Y,\eta)$$

$\rightsquigarrow$ regard $A : \Gamma(TM) \times \Gamma(TM) \times \Gamma(NM) \longrightarrow C^\infty(M)$

Define: $\forall\, X,Y,Z \in \Gamma(TM)$, $\forall\, \eta \in \Gamma(NM)$,

$$(\nabla_Z A)(X,Y,\eta) := Z\big(A(X,Y,\eta)\big) - A(\nabla_Z X, Y, \eta) - A(X, \nabla_Z Y, \eta) - A(X,Y,\nabla_Z^\perp \eta)$$

Thm: ("Constraint Equations" for isometric immersions)

The following equations hold for $M \subseteq (\bar{M}, \bar{g})$:

$\forall\, X,Y,Z,W \in \Gamma(TM)$, $\forall\, \eta, \xi \in \Gamma(NM)$. we have

Gauss: $$\bar{R}(X,Y,\underset{T}{\underset{=}{Z}},\underset{T}{\underset{=}{W}}) = R(X,Y,Z,W) - \langle A(Y,W), A(X,Z)\rangle + \langle A(X,W), A(Y,Z)\rangle$$

Ricci: $$\bar{R}(X,Y,\underset{N}{\underset{=}{\eta}},\underset{N}{\underset{=}{\xi}}) = \langle R^\perp(X,Y)\eta, \xi\rangle + \langle \underbrace{[S_\eta, S_\xi]}_{\substack{\text{Commutator} \\ S_\eta \circ S_\xi - S_\xi \circ S_\eta}}(X), Y\rangle$$

Codazzi: $$\bar{R}(X,Y,\underset{T}{\underset{=}{Z}},\underset{N}{\underset{=}{\eta}}) = (\nabla_Y A)(X,Z,\eta) - (\nabla_X A)(Y,Z,\eta)$$

Idea of Proof: $\exists$ orthogonal splitting $T\bar{M} = TM \oplus NM$

Proof: Recall: $\bar\nabla_X Y = \nabla_X Y + A(X,Y)$

: tangent to M

: normal to M

$$\bar R(X,Y)Z := \bar\nabla_Y \bar\nabla_X Z - \bar\nabla_X \bar\nabla_Y Z + \bar\nabla_{[X,Y]} Z$$

$$= \bar\nabla_Y\left(\nabla_X Z + A(X,Z)\right) - \bar\nabla_X\left(\nabla_Y Z + A(Y,Z)\right) + \nabla_{[X,Y]}Z + A([X,Y],Z)$$

$$= \nabla_Y\nabla_X Z + A(Y,\nabla_X Z) + \bar\nabla_Y\left(A(X,Z)\right) - \nabla_X\nabla_Y Z - A(X,\nabla_Y Z) - \bar\nabla_X\left(A(Y,Z)\right) + \nabla_{[X,Y]}Z + A([X,Y],Z)$$

$$= R(X,Y)Z + A(Y,\nabla_X Z) - A(X,\nabla_Y Z) + A([X,Y],Z) + \bar\nabla_Y(A(X,Z)) - \bar\nabla_X(A(Y,Z)).$$

Taking inner product with a tangential $W \in \Gamma(TM)$,

$$\bar R(X,Y,Z,W) = R(X,Y,Z,W) + \langle \bar\nabla_Y(A(X,Z)), W\rangle - \langle \bar\nabla_X(A(Y,Z)), W\rangle$$

$$= R(X,Y,Z,W) - \langle A(X,Z), \bar\nabla_Y W\rangle + \langle A(Y,Z), \bar\nabla_X W\rangle$$

Gauss!

$$= R(X,Y,Z,W) - \langle A(X,Z), (\bar\nabla_Y W)^N\rangle + \langle A(Y,Z), (\bar\nabla_X W)^N\rangle$$

$$= R(X,Y,Z,W) - \langle A(X,Z), A(Y,W)\rangle + \langle A(Y,Z), A(X,W)\rangle$$

Taking inner product with a normal $\eta \in \Gamma(NM)$,

$$\bar R(X,Y,Z,\eta) = A(Y,\nabla_X Z,\eta) - A(X,\nabla_Y Z,\eta) + A([X,Y],Z,\eta) + \underline{\langle \bar\nabla_Y(A(X,Z)),\eta\rangle} - \underline{\langle \bar\nabla_X(A(Y,Z)),\eta\rangle}$$

$$\left(\begin{aligned} \underline{\text{Note}}:\ & \langle \bar\nabla_Y(A(X,Z)), \eta \rangle \\ &= Y(A(X,Z,\eta)) - \langle A(X,Z), (\bar\nabla_Y \eta)^N \rangle \\ &= Y(A(X,Z,\eta)) - \langle A(X,Z), \nabla^\perp_Y \eta \rangle \end{aligned} \right)$$

$$\begin{aligned} \bar R(X,Y,Z,\eta) &= \underline{A(Y,\nabla_X Z,\eta)} - \underline{A(X,\nabla_Y Z,\eta)} + A(\underline{\nabla_X Y} - \underline{\nabla_Y X}, Z,\eta) \\ &\quad + \underline{Y(A(X,Z,\eta))} - \underline{A(X,Z, \nabla^\perp_Y \eta)} \\ &\quad - \underline{X(A(Y,Z,\eta))} + \underline{A(Y,Z,\nabla^\perp_X \eta)} \\ &= \underline{(\nabla_Y A)(X,Z,\eta)} - (\nabla_X A)(Y,Z,\eta) \end{aligned}$$

Codazzi!

$\underline{\text{Recall}}$: $\quad \bar\nabla_X \eta = -S_\eta(X) + \nabla^\perp_X \eta$

$$\begin{aligned} \bar R(X,Y)\eta &= \bar\nabla_Y \bar\nabla_X \eta - \bar\nabla_X \bar\nabla_Y \eta + \bar\nabla_{[X,Y]}\eta \\ &= \bar\nabla_Y(-S_\eta(X) + \nabla^\perp_X \eta) - \bar\nabla_X(-S_\eta(Y) + \nabla^\perp_Y \eta) \\ &\qquad - S_\eta([X,Y]) + \nabla^\perp_{[X,Y]}\eta \\ &= \text{“Tangential terms”} \\ &\qquad - [\bar\nabla_Y(S_\eta(X))]^N + [\bar\nabla_X(S_\eta(Y))]^N \\ &\qquad + \nabla^\perp_Y \nabla^\perp_X \eta - \nabla^\perp_X \nabla^\perp_Y \eta + \nabla^\perp_{[X,Y]}\eta \end{aligned}$$

Taking inner product with a normal $\zeta \in \Gamma(NM)$.

$$\begin{aligned} \bar R(X,Y,\eta,\zeta) &= \langle R^\perp(X,Y)\eta, \zeta \rangle \\ &\quad - \langle [\bar\nabla_Y(S_\eta(X))]^N, \zeta \rangle + \langle [\bar\nabla_X(S_\eta(Y))]^N, \zeta \rangle \end{aligned}$$

$$\left(\text{Note: } \quad \langle [\bar{\nabla}_Y (S_\eta(X))]^N, \xi \rangle = \langle \bar{\nabla}_Y (S_\eta(X)), \xi \rangle = Y(\underbrace{\langle S_\eta(X), \xi \rangle}_{=0}) - \langle S_\eta(X), (\bar{\nabla}_Y \xi)^T \rangle = \langle S_\eta(X), S_\xi(Y) \rangle \right)$$

$$\bar{R}(X,Y,\eta,\xi) = \langle R^\perp(X,Y)\eta, \xi \rangle + \langle S_\eta(X), S_\xi(Y) \rangle - \langle S_\eta(Y), S_\xi(X) \rangle$$

Ricci!

$$= \langle R^\perp(X,Y)\eta, \xi \rangle + \langle (S_\xi \circ S_\eta - S_\xi \circ S_\eta)(X), Y \rangle$$

∎

Several Remarks

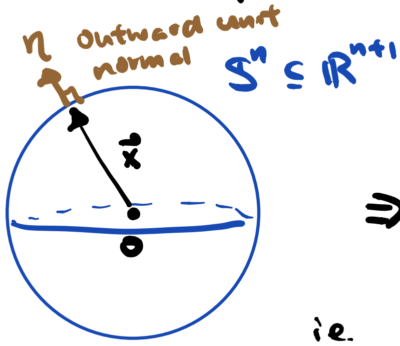
(1) In codimension 1 (ie. hypersurface), the Ricci equation is trivial.

Reason: codim 1 $\Leftrightarrow$ NM is 1-dim'l $\quad \xi = f\eta$

Ricci: $\underbrace{\bar{R}(X,Y,\eta,\eta)}_{=0} = \underbrace{\langle R^\perp(X,Y)\eta, \eta \rangle}_{=0} + \langle \underbrace{[S_\eta, S_\eta]}_{=0}(X), Y \rangle$

(2) The unit sphere $S^n \subseteq (\mathbb{R}^{n+1}, \bar{g}_{Eucl.})$ has $K \equiv 1$.

Pf:



$\eta = \vec{x}$ position vector

$$\Rightarrow \bar{\nabla}_X \eta = \bar{\nabla}_X \vec{x} = X \Rightarrow S_\eta(X) = -X$$

ie. $A_\eta(X,Y) = \langle S_\eta(X), Y \rangle = -\langle X, Y \rangle$

Gauss: $\underbrace{\bar{R}(X,Y,Z,W)}_{\substack{\equiv 0 \\ \because\ \mathbb{R}^{n+1} \text{ is flat}}} = R(X,Y,Z,W) - \langle A(Y,W), A(X,Z)\rangle + \langle A(X,W), A(Y,Z)\rangle$

$$\Rightarrow \quad R^{S^n}(X,Y,Z,W) = \langle Y,W\rangle\langle X,Z\rangle - \langle X,W\rangle\langle Y,Z\rangle$$

ie $K^{S^n} \equiv 1$.

(iii) If $M^{n-1} \subset (\bar{M}^n, \bar{g})$ codim 1 and $(\bar{M}, \bar{g})$ has constant sectional curvature, then for η = unit normal (locally)

Codazzi eq^{n} $\Leftrightarrow$ $\nabla_X(S_\eta(Y)) - \nabla_Y(S_\eta(X)) = S_\eta([X,Y])$

(Ex: Prove this.)

Summary: Submanifold $M^k \subseteq (\bar{M}^n, \bar{g})$, $\bar{g}|_M =: g$

$\rightsquigarrow$ Study (M^k, g) from two perspectives: intrinsic OR extrinsic

Roughly speaking: $T_p\bar{M} = T_pM \oplus (T_pM)^\perp$

"differentiate g" $\rightsquigarrow$ $\quad \bar{\nabla} = \nabla^T + \bar{\nabla}^N$

Riem. conn. on $(\bar{M}, \bar{g})$; Riem. conn. on (M, g) ; 2^{nd} f.f. $A(X,Y) := (\bar{\nabla}_X Y)^N$

Equivalently, $S_\eta(X) := -(\bar{\nabla}_X \eta)^T$, $\eta \in \Gamma(NM)$

"differentiate g twice" $\rightsquigarrow$ $\exists$ "constraint eq^{n}"

Note: $\langle A(X,Y), \eta\rangle = \langle S_\eta(X), Y\rangle$

Gauss: $\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) - \langle A(Y,W), A(X,Z)\rangle + \langle A(X,W), A(Y,Z)\rangle$

Codazzi: $\bar{R}(X,Y,Z,\eta) = (\nabla_Y A)(X,Z,\eta) - (\nabla_X A)(Y,Z,\eta)$

Ricci: $\bar{R}(X,Y,\eta,\zeta) = \langle R^{\perp}(X,Y)\eta, \zeta\rangle + \langle [S_\eta, S_\zeta](X), Y\rangle$

Consider the special case of $M^2 \subset (\mathbb{R}^3, g_{Eucl.})$

Fix $p \in M$, $\sigma = T_pM \subseteq T_p\mathbb{R}^3$ $\quad \sigma = \mathrm{Span}\{e_1, e_2\}$ O.N.B.

$$\bar{R}(e_1,e_2,e_1,e_2) = R(e_1,e_2,e_1,e_2) - \langle A(e_2,e_2), A(e_1,e_1)\rangle + \langle A(e_1,e_2), A(e_2,e_1)\rangle$$

($\because \mathbb{R}^3$ is flat)

$$\Rightarrow \quad 0 = R_{1212} - A_{22}A_{11} + A_{12}^2$$

Gauss' Golden Theorem!

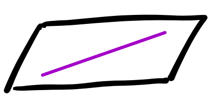
ie $\quad R_{1212} = A_{11}A_{22} - A_{12}^2 = \det\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix} =: K$ Gauss curvature

Def^n: $M^k \subseteq (\bar{M}^n, \bar{g})$ is totally geodesic if $A \equiv 0$ at every $p \in M$

Eg.) $(\mathbb{R}^n, g_{Eucl.})$

$\cup\!|$

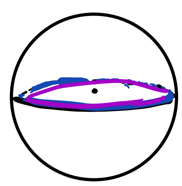
$\mathbb{R}^{n-1}$



totally geodesic hypersurface

(S^n, g_{round})

$\cup\!|$ totally geodesic hypersurface



$\leftarrow S^{n-1}$

Prop: $M^k \subseteq (\bar{M}^n, \bar{g})$ totally geodesic $\Leftrightarrow$ every geodesics in M are geodesics in $\bar{M}$.

Pf: For any smooth curve $\gamma : I \to M$,

$$\bar{\nabla}_{\gamma'}\gamma' = (\bar{\nabla}_{\gamma'}\gamma')^T + (\bar{\nabla}_{\gamma'}\gamma')^N$$

$$= \nabla_{\gamma'}\gamma' + \underbrace{A(\gamma',\gamma')}_{\equiv 0}$$

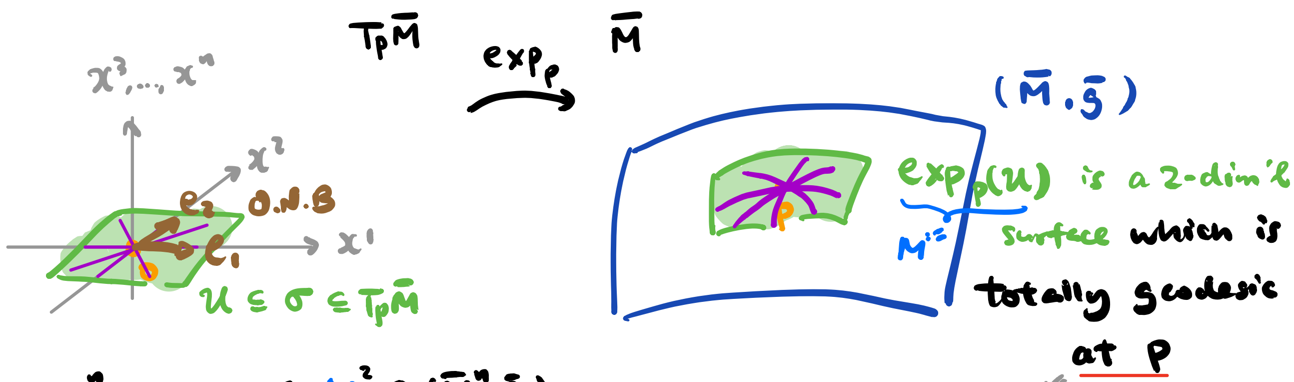
$\because$ totally geodesic

Note: Geodesics in $\bar{M}$ lying inside M are always geodesic in M

$\blacksquare$

This gives a geometric interpretation of "Sectional curvature" in terms of "Gauss curvature" for surfaces.

Recall: In geodesic normal coord. centered at $p \in (\bar{M}^n, \bar{g})$.



By Gauss eq$^{\underline{n}}$, at $p \in M^2 \subseteq (\bar{M}^n, \bar{g})$.

$$\underbrace{\bar{R}(e_1, e_2, e_1, e_2)}_{\bar{K}_p(\sigma)} = \underbrace{R(e_1, e_2, e_1, e_2)}_{\substack{\| \text{ Gauss curvature} \\ K_M(p)}} + \overbrace{(\text{quadratic terms of } A)}^{=0 \text{ at } p}$$

Remark: Totally geodesic submanifolds rarely exists in general $(\bar{M}^n, \bar{g})$.

We want to define a "weaker" notion.

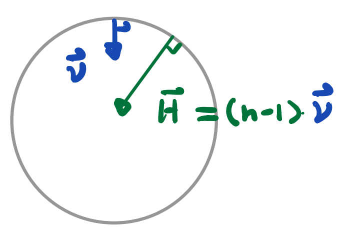
Def$^{\underline{n}}$: $M^k \subseteq (\bar{M}^n, \bar{g})$ ⇝ The mean curvature vector at $p \in M$

$$\vec{H}(p) := \sum_{i=1}^{k} A_p(e_i, e_i) \quad \text{where } \{e_1, \dots, e_k\} \text{ O.N.B. for } T_pM$$

And we say M is minimal if $\vec{H} \equiv \vec{0}$ at every $p \in M$

Remarks: 1) totally geodesic $\overset{\Longrightarrow}{\not\Longleftarrow}$ minimal

E.g.) $S^{n-1} \subseteq \mathbb{R}^n$



2) In codim. 1 case, we write

$$\vec{H} = H \vec{\nu}$$

H: (scalar) mean curvature

$\vec{\nu}$: unit normal

(Sign depends on choice of $\vec{\nu}$)

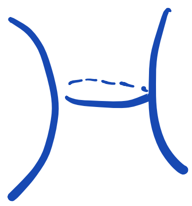
3) When $k = \dim M = 1$, then "minimal" $\Leftrightarrow$ "geodesic"

In fact, minimal k-submanifolds are critical points to the k-dim'l area functional, just like "geodesics" are critical points to the length functional.

E.g.) Minimal surfaces in $\mathbb{R}^3$

plane

totally geodesic

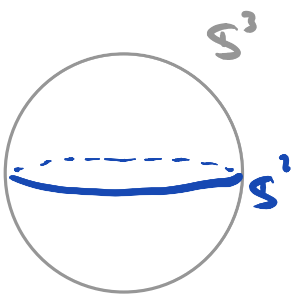


catenoid

helicoid

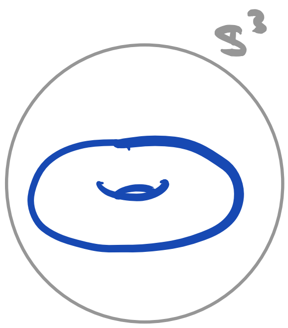
∞'ly many more ......

Minimal surfaces in (S^3, g_{round})



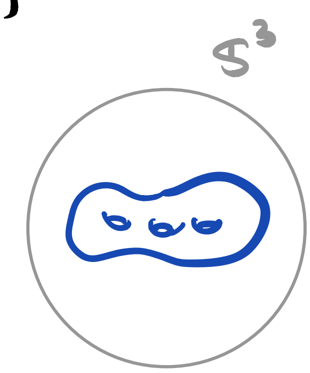
great sphere

totally geodesic



Clifford torus

(HW)



Lawson surfaces

(~1970's)

∞'ly many ......